Open problems in the spectral theory of signed graphs

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Based on the work by Belardo et al. "Open problems in the spectral theory of signed graphs." arXiv preprint arXiv:1907.04349 (2019).

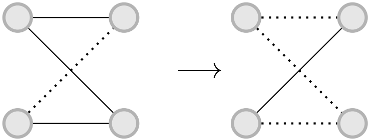
Sign-symmetric graphs

Sign-symmetric graphs

Theorem

A graph is bipartite if and only if its adjacency spectrum is symmetric with respect to the origin.

Bipartite signed graphs have an interesting property: they are switching equivalent to their negation. $(G, \sigma) \sim (G, -\sigma)$.



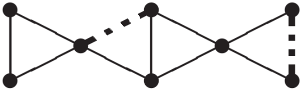
Property

This property is known as sign-symmetry.

Sign-symmetric graphs

Definition

A signed graph $\Gamma = (G, \sigma)$ is said to be sign-symmetric if it is switching equivalent to its negation $-\Gamma = (G, -\sigma)$.

This property is not exclusive to bipartite graphs: 

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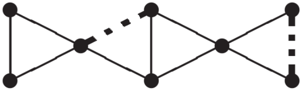
This property is not exclusive to bipartite graphs: 

Let Γ be sign-symmetric. Is its adjacency spectrum symmetric? Recall that the spectrum is invariant under switching...

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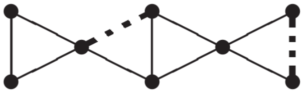
This property is not exclusive to bipartite graphs: 

Let Γ be sign-symmetric. Is its adjacency spectrum symmetric? Recall that the spectrum is invariant under switching...

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Let Γ be a sign-symmetric graph. Then its adjacency spectrum is symmetric with respect to the origin.

Sign-symmetric graphs

Question

Are there signed graphs whose spectrum is symmetric with respect to the origin but are not sign-symmetric?

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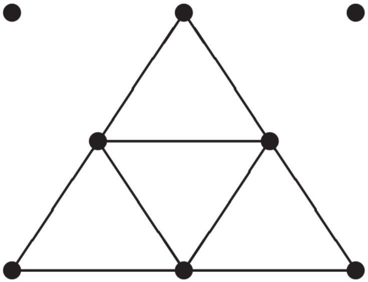
Seidel matrix: $S(G) = J - I - 2A$.

Example from (Et-Taoui and Fruchard, 2018)



Sign-symmetric graphs

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Are there ***non-complete*** *connected signed graphs whose spectrum is symmetric with respect to the origin but are not sign-symmetric?*

Sign-symmetric graphs

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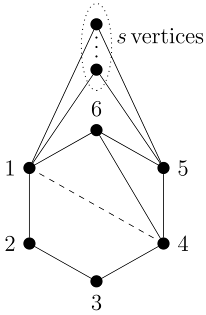
Solved: (Ghorbani et al., 2020)



The number of distinct eigenvalues

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Let us take a look at unsigned graphs first[1].

Theorem

Let G have a diameter of d. The number of distinct adjacency eigenvalues of G is at least $d + 1$.

This result is not true for signed graphs.

Question for you

What is the only unsigned graph with exactly two distinct eigenvalues?

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Yes, K_n!

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Let G be connected and regular. Then it has three distinct eigenvalues if and only if it is strongly regular[2].

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Are there non-strongly regular, non-complete bipartite graphs with exactly three eigenvalues? Yes. (van Dam, 1998)

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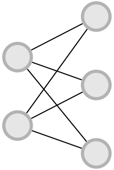
$K_{2,3}$: 

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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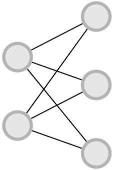

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$$\lambda = (\sqrt{6}, 0, 0, 0, -\sqrt{6}).$$

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The number of distinct eigenvalues

Problem

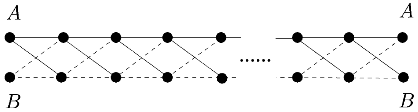
Does a graph with three distinct eigenvalues have at most three distinct degrees?

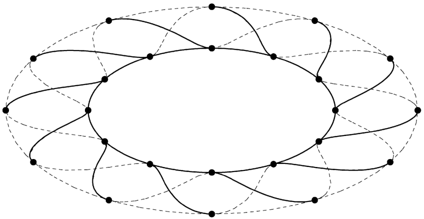
Negative answer for graphs with four and five distinct eigenvalues (Van Dam et al., 2014).

The number of distinct eigenvalues

Back to signed graphs. Recall that in unsigned graphs, # of distinct eigenvalues $>$ diameter. A signed counterexample, with diameter $\lfloor \frac{k}{2} \rfloor$, two distinct eigenvalues:

(McKee and Smyth, 2007)





The number of distinct eigenvalues

Therefore, the answer to the next question is not easy for signed graphs:

Problem

Characterize all connected signed graphs whose spectrum consists of two distinct eigenvalues.

Definition

Two vertices are at signed distance k if they are at distance k and the difference between the numbers of positive and negative walks of length k among them is nonzero. Otherwise the signed distance is set to 0. The maximum signed distance is the signed diameter $diam^{\pm}(\Gamma)$.

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Theorem

Let Γ be a connected signed graph with m distinct eigenvalues. Then $m \geq diam^{\pm}(\Gamma) + 1$.

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- $\pm K_n$
- Huang's hypercube for the Sensitivity Conjecture! (Huang, 2019)

Signature minimizing the spectral radius

Signature minimizing the spectral radius

Let $\rho(\Gamma) = \max_i\{|\lambda_i(\Gamma)|\}$.

Problem

Let Γ be a simple and connected unsigned graph. Determine the signature $\bar{\sigma}$ such that for any signature σ of Γ, we have $\rho(\Gamma, \bar{\sigma}) \leq \rho(\Gamma, \sigma)$.

This problem is very important! Let's see why.

Definition

A d-regular graph G is a Ramanujan graph if $\max\{|\lambda_2|, |\lambda_n|\} \leq 2\sqrt{d-1}$.

Signature minimizing the spectral radius

Definition

Consider a signed graph Γ. The 2-lift of Γ is an unsigned graph $\Gamma' = (V \times \{+1, -1\}, E)$ where (x, s) is adjacent to $(y, s\sigma(xy))$, for $s = \pm 1$.

Signature minimizing the spectral radius

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Consider a signed graph Γ. The 2-lift of Γ is an unsigned graph $\Gamma' = (V \times \{+1, -1\}, E)$ where (x, s) is adjacent to $(y, s\sigma(xy))$, for $s = \pm 1$.

Theorem

Let G be the underlying graph of Γ. The spectrum of Γ' is the union of the spectra of G and Γ.

Proof: The adjacency matrix of Γ' is $A_{\Gamma'} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$, where A_1 (resp. A_2) is the adjacency matrix of $(V, s) \times (V, s)$ (resp. $(V, s) \times (V, -s)$), where $s = \pm 1$.

Signature minimizing the spectral radius

Theorem

Every connected d-regular graph has a signing with spectral radius at most $c\sqrt{d \log^3 d}$, where $c > 0$ is some absolute constant. (Bilu and Linial, 2006)

Conjecture

Every connected d-regular graph has a signing with spectral radius at most $2\sqrt{d-1}$. (Bilu and Linial, 2006)

Theorem

Let G be a connected d-regular graph. Then there exists a signature σ of G such that the largest eigenvalue of A_σ is at most $2\sqrt{d-1}$. (Marcus et al., 2013)

Signature minimizing the spectral radius

Problem

Let Γ be a simple and connected unsigned graph. Determine the signature $\bar{\sigma}$ such that for any signature σ of Γ, we have $\rho(\Gamma, \bar{\sigma}) \leq \rho(\Gamma, \sigma)$.

"As an amusing exercise, we challenge the readers to solve this problem by finding a signature of the Petersen graph or of their favorite graph that minimizes the spectral radius." (Belardo et al., 2019)

Spectral determination

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Are unsigned graphs determined by their spectrum (up to isomorphism)?

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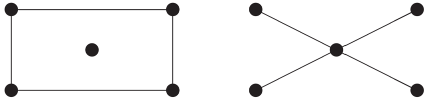
No Van Dam and Haemers (2003)



Spectral determination

Some are, such as the path P_n and the cycle C_n.

But of course, not the case for signed graphs (Akbari et al., 2018a,b).

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But of course, not the case for signed graphs (Akbari et al., 2018a,b).

Theorem

The signed path P_n is determined by its spectrum if and only if $n \equiv 0, 1, 2 \pmod 4$ unless $n \in \{8, 13, 14, 17, 29\}$, or $n = 3$.

Theorem

- *Odd signed cycles C^+_{2n+1}, C^-_{2n+1} and C^-_4 are determined by their spectrum.*
- *Even signed cycles C^+_{2n}, C^-_{2n} except C^-_4 are not determined by their spectrum.*

Spectral determination

Proposition

From the eigenvalues of a signed graph Γ we obtain the following invariants:

- *number of vertices and edges*
- *the difference between the number of positive and negative triangles:* $\frac{1}{6}\sum_i \lambda_i^3$;
- *the difference between the number of positive and negative closed walks of length p:* $\sum_i \lambda_i^p$.

Other problems

- Characterize graphs with bounded largest/smallest eigenvalue.
- Hoffman program and Hoffman graphs.
- Conference matrices.
- Seidel matrices.

Thanks!

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