Alexander Grothendieck and some inapproximable matrix norms

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Some inapproximable norms

This is the work of Alon and Naor (2004).

Definition

Cut norm: given an $n \times m$ matrix $A = (a_{ij})$,

$$\|A\|_C = \max_{I \subseteq [n], J \subseteq [m]} \left| \sum_{i \in I, j \in J} a_{ij} \right| .$$

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More generally,

$$\|A\|_{p \to q} = \max_{x \neq 0} \frac{\|Ax\|_q}{\|x\|_p} .$$

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Indeed, if $x_i, y_j \in \{-1, 1\}$,

$$\sum_{ij} a_{ij}x_i y_j = \sum_{i:x_i=1,j:y_j=1} a_{ij} - \sum_{i:x_i=1,j:y_j=-1} a_{ij} - \sum_{i:x_i=-1,j:y_j=1} a_{ij} + \sum_{i:x_i=-1,j:y_j=-1} a_{ij}.$$

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On the other hand, suppose $\|A\|_C = \sum_{i\in I, j\in J} a_{ij}$. Let $x_i = 1$ if $i \in I$, $x_i = -1$ otherwise (same for y_j, J).

$$\begin{aligned} \|A\|_C &= \sum_{i,j} a_{ij} \frac{1+x_i}{2}\frac{1+y_i}{2} \\ &= \frac{1}{4}\sum_{i,j} a_{ij} + \frac{1}{4}\sum_{i,j} a_{ij}x_i + \frac{1}{4}\sum_{i,j} a_{ij}x_j + \frac{1}{4}\sum_{i,j} a_{ij}x_iy_j. \end{aligned}$$

Inapproximability

Computing $\|A\|_C$ or $\|A\|_{\infty\to 1}$ is MAXSNP-hard (no PTAS).

Proposition

Given a (weighted or unweighted) graph $G = (V, E)$, there is an efficient way to construct a real $2|E|$ by $|V|$ matrix A, such that

$$MAXCUT(G) = \|A\|_C = \|A\|_{\infty\to 1}/4.$$

Therefore, the problems of computing $\|A\|_C$ or $\|A\|_{\infty\to 1}$ are both MAXSNP-hard.

Proof: Orient G arbitrarily. For each $1 \le i \le |E|$, if e_i is oriented from v_j to v_k, $a_{2i-1,j} = a_{2i,k} = 1$ and $a_{2i-1,k} = a_{2i,j} = -1$. The rest of A is 0.

$MAXCUT = \|A\|_C = \|A\|_{\infty\to 1}/4$. □

So we will try to approximate $\|A\|_{\infty\to 1}$.

Note

$$\|A\|_{\infty\to 1} = \max_{x,y} \quad \sum_{i,j} a_{ij} x_i y_j$$
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so we can relax:

$$\max_{\{u_i\},\{v_j\}} \quad \sum_{i,j} a_{ij} \langle u_i, v_j \rangle$$

$$\text{s.t.} \quad \|u_i\| = \|v_j\| = 1 \text{ for all } i, j,$$

We will use MAX_{SDP} for the maximum of the SDP relaxation.

The Grothendieck inequality

Theorem

Grothendieck inequality. *There is a constant* $K_{\mathbb{R}}$ *such that for any matrix* A,

$$MAX_{SDP} \leq K_{\mathbb{R}} \|A\|_{\infty \to 1}.$$

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Let's prove that

$$K_{\mathbb{R}} \le \frac{\pi}{2\ln(1+\sqrt{2})},$$

shall we?

Lemma

Grothendieck identity. *Let u, v be unit vectors in a Hilbert space H. If z is a randomly picked unit vector in H, then*

$$\mathbb{E}[sign(\langle u, z\rangle)sign(\langle v, z\rangle)] = \frac{2}{\pi}arcsin(\langle u, v\rangle).$$

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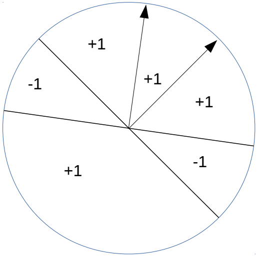

The previous result is nice, because with straightforward rounding, we get

$$\mathbb{E}\left[\sum_{i,j} a_{ij} x_i y_j\right] = \sum_{i,j} a_{ij} \frac{2}{\pi} arcsin(\langle u_i, v_j \rangle).$$

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So how nice would it be if we could find vectors u_i', v_j' such that

$$arcsin(\langle u_i', v_j' \rangle) = c\langle u_i, v_j \rangle,$$

for some constant c...

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$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

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Consider the vectors $T(u), S(v)$ defined by the following sequence:

$$T(u)_k = (-1)^k \sqrt{\frac{c^{2k+1}}{(2k+1)!}} u^{\otimes(2k+1)}$$

$$S(v)_k = \sqrt{\frac{c^{2k+1}}{(2k+1)!}} v^{\otimes(2k+1)}$$

These vectors are in the Hilbert space $\oplus_{k=0}^{\infty} H^{\otimes(2k+1)}$, where $u, v \in H$.

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We have $\langle T(u), S(v)\rangle = \sin(c\langle u_i, v_j\rangle)$.

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Hint: $\sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$.

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So we choose c so that $\sinh(c) = 1 \Leftrightarrow c = \text{arcsinh}(1) = \log(1 + \sqrt{2})$.

We now choose a random unit vector $z \in \oplus_{k=0}^{\infty} H^{\otimes(2k+1)}$ and set $x_i = sign(\langle u_i', z\rangle)$, $y_j = sign(\langle v_j', z\rangle)$. We have

$$\mathbb{E}\left[\sum_{i,j} a_{ij} x_i y_j\right] = \frac{2}{\pi}\sum_{i,j} a_{ij} \arcsin(\langle u_i', v_j'\rangle) = \frac{2\log(1+\sqrt{2})}{\pi}\sum_{i,j} a_{ij}\langle u_i, v_j\rangle.$$

This establishes

$$K_{\mathbb{R}} \leq \frac{\pi}{2\log(1+\sqrt{2})}.$$

Proof due to Krivine[1].

[1] `https://www.irif.fr/~krivine/`

Algorithmic applications

We have established an upper bound of Grothendieck's constant by showing that there exists a convenient set of vectors $\{u_i', v_j'\} \subset \bigoplus_{k=0}^{\infty} H^{\otimes(2k+1)}$.

This suggests an algorithm to approximate $\|A\|_{\infty\to 1}$.

1. Relax $\|A\|_{\infty\to 1}$.
2. Solve the SDP.
3. Find $\{u_i', v_j'\}$.
4. Choose z randomly and round by setting $x_i = sign(\langle u_i', z\rangle)$, $y_j = sign(\langle v_j', z\rangle)$.

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No problem. $\{u'_i, v'_j\}$ are $m + n$ vectors, so we can find a valid set in $(m + n)$-dimensional space. How?

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Add 1 row and 1 column to A to obtain A':

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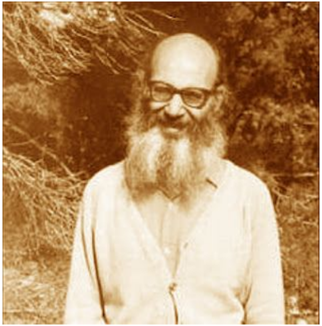
It is

$$\|A\|_C = \|A'\|_C = \frac{1}{4}\|A\|_{\infty\to 1}.$$

The last equality is because rows and columns sum to zero.

This is just one of four approaches in the paper by Alon and Naor (2004). Check it out!

Alexander Grothendieck (1928 - 2014)



Thanks!

References I

Alon, N. and Naor, A. (2004). Approximating the cut-norm via grothendieck's inequality. In *Proceedings of the thirty-sixth annual ACM symposium on Theory of computing*, pages 72–80.